Hydroelectric Power

Water, falling as rain (or snow) on high hills or mountains, has potential energy that can be harvested as useful energy if the water can be diverted through a suitable system of pipes or conduits as it flows from the hills or mountains to a lower level. This is the source of hydroelectric power: potential enery is converted to mechanical energy, and for electricity generation, mechanical energy is converted to electrical energy. In these exercises, we will look at some of the factors involved in converting potential energy of water into mechanical energy, deriving some elementary principles and scaling, and look at some calculations of how much energy can be harvested.

We start with Newton's second law, force is mass times acceleration, and look at the units,

$$\text{Force} \equiv \text{Mass} \ \times \frac{\text{Length}}{\text{Time}^2},$$

and rearrange

$$\text{Force} \equiv \frac{\text{Mass}}{\text{Time}} \ \times \frac{\text{Length}}{\text{Time}},$$

and if this in interpreted for a liquid flowing with volumetric rate Q (m^3/s), density $\rho(\text{kg/m}^3)$ so that the mass flow rate is ρQ (kg/s), and where there is a velocity change, which we can denote Δv (m/s), then the force, F (Newtons) associated with the velocity change will be

$$F = \rho Q \Delta v.$$

This result, which you may already have studied, will be at the heart of calculations we will need to determine the conversion of potential energy into mechanical energy in fluid flow.

Question: Estimate the force in Newtons on a wing mirror that is outside of a vehicle travelling at 80km/h, assuming that any air impacting on the wing mirror is deflected only sideways and friction is neglected. Take the density of air as approximately $1.225\ \text{kg/m}^3$. *Again, neglecting viscous effects, estimate the force on your body if you were to try to run at* $1\ \text{m/s}$ *through waist high water, assuming that any deflected water is just pushed sideways. What conclusion might you draw from the different magnitudes of the these forces?*

Having seen that for a steadily flowing fluid, a change in velocity requires or results in a force (depending on whether the fluid speeds up or slows down), we can now look at one of the oldest means of converting gravitational potential energy into mechanical energy: the water wheel, and in particular, a modern version used in hydroelectric power generation called the Pelton wheel. The layout is illustrated in the two diagrams, and many pictures are available on the web if you search for 'Pelton wheel'. A jet of water is directed at specially shaped buckets on a wheel, the change in fluid velocity results in a force that causes the wheel to rotate, generating mechanical torque that can drive an electric generator. The buckets are shaped so that fluid flows smoothly around the inside shape of the bucket.

Question: why is the bucket symmetric about the centre of the jet?

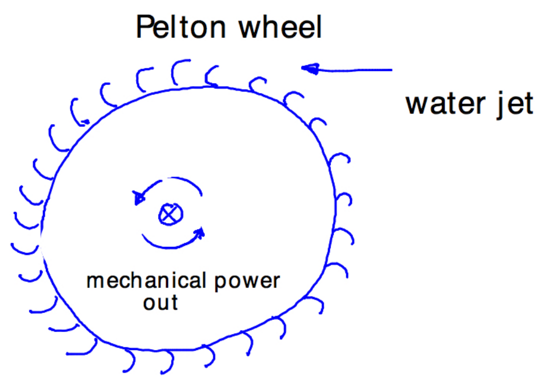


Figure 1: Schematic for Pelton wheel.

We use an idealised model that assumes that if the fluid enters the centre part of the bucket with speed V_1, and moves without friction around the bucket to leave in the direction of the outer rim of the bucket with the same speed, V_1. In the calculations here, we will assume that the bucket is designed to turn the water flow direction through 180 degrees so that the change in velocity across the bucket is

$$\Delta v = 2V_1.$$

In order to get rotational mechanical energy from the wheel, we will need to allow the wheel to rotate: suppose the wheel radius is R (m) and the rotational speed is ω (radians/s).

Question: Prove that the speed of the bucket is

$$U = \omega R.$$

As a hint, consider the time taken for one complete rotation and the distance travelled in that time.

Including the bucket speed is important since as the bucket is moving away from the water jet source, the relative speed of fluid approaching the bucket is less than the absolute speed of the water jet. If we take the absolute speed to be V, and the bucket speed U, then the relative speed of water with the bucket is only

$$V_1 = V - U,$$

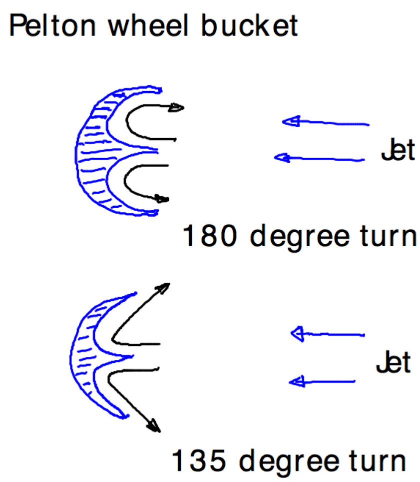


Figure 2: Pelton wheel buckets

and the change in fluid velocity when the fluid enters with speed V_1 and leaves with speed V_1 in the *opposite* direction, is

$$\Delta v = 2(V - U).$$

Hence the force on the Pelton wheel will be

$$F = \rho Q \Delta v = 2\rho Q(V - U).$$

Next we need to recall units and that

$$\text{Work} = \text{Force} \ \times \ \text{Distance},$$

$$\text{Power} = \frac{\text{Work}}{\text{Time}} = \text{Force} \ \times \ \frac{\text{Distance}}{\text{Time}},$$

or as you may have already studied,

$$\text{Power} = \text{Force} \ \times \ \text{Velocity}.$$

In modelling the Pelton wheel, we neglect that there is only a finite number of buckets and that the jet hitting the buckets is not truly steady and assume that the force on the rim of the wheel is constant and uniform as the wheel rotates. In that case the power of the Pelton wheel, denoted P (Watts) will be

$$P = 2\rho Q(V - U) \times U.$$

The final link needed is to assume that the jet of water is of area A, so that $Q = VA$, and

$$P = 2\rho AVU(V - U).$$

This is a very interesting result: when the wheel is not turning, $U = 0$, and obviously there is no power transmitted to the generator. However, if the wheel turns too fast and $U = V$, then again there is no power generated, since the speed of the water passing through the wheel is unaltered. The force on the wheel is maximum when the wheel is not rotating, and the force decreases to zero when the wheel rotates with rim speed V.

Question: For any arbitrary value V, sketch the function

$$f(U) = U(V - U).$$

Deduce that the power, P, will have a maximum value when $U = V/2$. Give a physical explanation as to why the power is maximum for this particular wheel speed.

Using this understanding of how an idealised Pelton wheel generates power, the last step is to look at the potential energy changes as water falls over some height, let's assume the height is H (m). Consider the mass, m (kg), of water that would flow in a time T (seconds),

$$m = \rho QT,$$

and if this mass fell a height H starting from rest, and emerged with velocity V, then conversion of potential energy to kinetic energy would, with gravitational constant g, give

$$\frac{1}{2}mV^2 = \frac{1}{2}\rho QTV^2 = mgH = \rho QTgH,$$

and cancelling terms in the second and fourth expression, and a little rearrangement,

$$V = \sqrt{2gH}.$$

Question: Show that in the ideaised situation where $U = V/2$, then the power output from the Pelton wheel would be

$$P = \sqrt{2}\rho A(gH)^{3/2}.$$

Check that this expression has the correct units. Calculate the ideal power produced if the jet area was 1 m^2 *and the water drop was* 300 m.

Question [harder]: We assumed that the fluid speed did not change as it moved around the bucket, one might expect viscous friction to reduce the speed of the fluid. Suppose the speed at the exit from the bucket was a given fraction of the inlet velocity, so that if $V_{in} = V_1$, then $V_{out} = \alpha V_1$ where α is a constant less than one. Work out the ideal power output for a Pelton wheel with buckets having a 180 degree fluid turn, and show that this will be reduced from the calculation above by a factor $(1 + \alpha)/2$.

If you are interested in hydroelectric power generation then there are many relevant web pages, just search for hydroelectric power or Pelton wheel, there are also many different water turbines, for example search for 'Kaplan turbine'.

Answers to Questions

Question: *Estimate the force in Newtons on a wing mirror that is outside of a vehicle travelling at 80km/h, assuming that any air impacting on the wing mirror is deflected only sideways and friction is neglected. Take the density of air as approximately* 1.225 kg/m^3. *Again, neglecting viscous effects, estimate the force on your body if you were to try to run at* 1 m/s *through waist high water, assuming that any deflected water is just pushed sideways. What conclusion might you draw from the different magnitudes of the these forces?*

We need to estimate the area of the wing mirror, there is no exact size, my estimate is around 10cm by 10cm, so area $A = 0.01$ m^2 but any sensible value is fine, although it will alter the calculations next. If the car is travelling at 80km/h, then as the air is pushed sideways, the change in velocity in the direction of travel will be the same as the vleocity, V, $\Delta v = 80000/3600$ m/s = 22.2 m/s and the air flow rate will be $Q = AV = 0.01 \times 22.2$ m^3/s = 0.222 m^3/s. Combining these with the given air density,

$$F = 1.225 \times 0.222 \times 22.2 = 6.04 \text{ N}.$$

Turning to the water flow, estimate area (again with no definite value) as say 100cm by 20cm, area $A = 0.2$ m^2, the flow rate will be $Q = AV = 0.2 \times 1 =$ 0.2 m^3/s, and Δv will also be the same as the velocity, and the density of water is 10^3 kg/m^3 so that

$$F = 10^3 \times 0.2 \times 1 = 200 \text{ N}.$$

One important observation we can make is that because of the very large density difference, the energy of a volume of moving water is around eight hundred that of a similar volume of moving air at the same speed, so devices that extract energy from moving water are likely to be physically much smaller than devices that extract energy from air (hence wind turbines usually have a very large diameter).

Question: *why is the bucket symmetric about the centre of the jet?*

The buckets are designed with sideways symmetry so that there is no resultant sideways force which would cause wear on the drive shaft bearing.

Question: *Prove that the speed of the bucket is*

$$U = \omega R.$$

As a hint, consider the time taken for one complete rotation and the distance travelled in that time.

If the angular speed is ω radians/second, then the buckt will take a time $T = 2\pi/\omega$ seconds to complete one rotation, during which it has travelled the perimeter of the wheel, $2\pi R$ metres, so assuming constant speed, the speed at the perimeter has to be

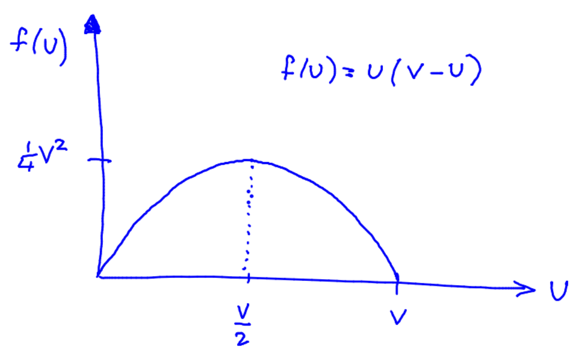
$$U = \frac{2\pi R}{\frac{2\pi}{\omega}} = \omega R.$$

Question: *For any arbitrary value V, sketch the function*

$$f(U) = U(V - U).$$

Deduce that the power, P, will have a maximum value when $U = V/2$. Give a physical explanation as to why the power is maximum for this particular wheel speed.

The function $f(U) = U(V - U)$ is a parabola, we know $f(0) = 0$, and $f(V) = 0$. It is also true that there is symmetry since $f(V - U) = f(U)$ so the point of symmetry is at $U = V - U$ or $U = V/2$. If you have studied calculus then you will already know how to show with calculus that $U = V/2$ is a stationay point and a maximum, but this is also easy to see from a sketch.



The importance of this wheel speed comes from looking at the velocity of the water relative to the jet after it passes through the bucket. If $U < V/2$ then the water will be moving backward towards the jet and have kinetic energy due to this velocity. If the wheel speed has $U > V/2$, the water will still be moving away from the jet, and again, will have kinetic energy that has not been used to drive the wheel. It is only when $U = V/2$ that the water has no velcity either towards or away from the jet, and in effect, just falls to the floor so that all the kinetic energy of the jet has been harvested.

Question: *Show that in the ideaised situation where $U = V/2$, then the power output from the Pelton wheel would be*

$$P = \sqrt{2}\rho A(gH)^{3/2}.$$

Check that this expression has the correct units. Calculate the ideal power produced if the jet area was 1 m^2 *and the water drop was* 300 m.

Using $U = V/2$, the power becomes

$$P = 2\rho AV \times \frac{V}{2} \times \frac{V}{2} = \frac{1}{2}\rho AV^3,$$

and then imserting $V = \sqrt{2gH}$,

$$P = \frac{1}{2}\rho A(2gH)^{3/2} = \sqrt{2}\rho A(gH)^{3/2}.$$

We can check the units

$$RHS \equiv \frac{\text{Mass}}{\text{Length}^3}\ \text{Length}^2\ (\frac{\text{Length}}{\text{Time}^2}\ \text{Length})^{3/2},$$

and collecting terms

$$RHS \equiv \frac{\text{Mass}}{\text{Length}}\ (\frac{\text{Length}}{\text{Time}})^3,$$

$$RHS \equiv \frac{\text{Mass} \times \text{Length}}{\text{Time}^2}\ \frac{\text{Length}}{\text{Time}},$$

and finally

$$RHS \equiv \text{Force} \times \text{Velocity},$$

so that the units are consistent.

In the example. the jet area may be thought large but of course multiple jets and multiple turbines may be used so individual jets may have smaller area. For the calculation,

$$P = \sqrt{2} \times 10^3 \times 1 \times (9.81 \times 300)^{3/2} = 225.8 \text{ Mw}.$$

The flow rate for this example is near 77 m^3/s.

Question [harder]: We assumed that the fluid speed did not change as it moved around the bucket, one might expect viscous friction to reduce the speed of the fluid. Suppose the speed at the exit from the bucket was a given fraction of the inlet velocity, so that if $V_{in} = V_1$, then $V_{out} = \alpha V_1$ where α is a constant less than one. Work out the ideal power output for a Pelton wheel with buckets having a 180 degree fluid turn, and show that this will be reduced from the calculation above by a factor $(1+\alpha)/2$.

The theory for this situation is virtually the same as that we have already seen, particularly if you have a good picture in your mind of how the Pelton wheel works. Here fluid moving (say) right to left with speed V_1 (velocity V_1) is then turned to move left to right with speed αV_1 (velocity $-\alpha V_1$), so the the change in *velocity* in the right to left direction is $(1+\alpha)V_1$. This changes the power produced to

$$P = (1+\alpha)\rho AVU(V-U).$$

and the change in speed around the bucket results in a factor $(1+\alpha)/2$ in the maixmum power.