Fun and games at the circus

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Problem: Imagine the following two-player video game. One player takes the role of a moody circus lion, the other plays its nimble tamer. Both players are confined to the circus arena, which has unit radius, and we think of both the lion and the tamer as points in the mathematical sense rather than spatially extended objects. As the game is about to begin the lion is performing a trick at the centre of the arena and the lion tamer stands at a safe distance of 0.5 from the lion. The lion's mood suddenly turns, and the game begins. Now both players move at the same constant speed in a direction of their choice. The lion's aim is to capture the lion tamer, the lion tamer's aim is to escape for as long as possible.

Does either player have a winning strategy? That is to say, is there a strategy for the lion which guarantees capture, or is there a strategy for the lion tamer which guarantees indefinite survival?

Solution: This problem was posed by R. Rado in the mid-1920s and became widely known after Littlewood discussed it in his *Miscellany* of 1953. It is tempting to think that the lion has the edge, because the best thing the lion tamer can do, surely, is to dash towards the perimeter and then move along it, in which case the lion wins by approaching the tamer while remaining on the straight line from the centre of the arena to the tamer. Can you see why the lion wins in this case?

This 'solution' was believed to be correct for around 25 years, until A.S. Besicovitch in the early 1950s realised that the argument is flawed. Indeed, the lion tamer can do rather better than to dash for the perimeter, and in fact Besicovitch showed that it is the tamer who has a winning strategy. This strategy requires the lion tamer to run in straight lines from its initial position T_0 to a new position T_1, from there to T_2 and so on, where $T_1, T_2, T_3, \dots$ are chosen in a particular way. Let us write O for the centre of the arena and L_k for the position of the lion at stage $k \geq 0$. Thus $L_0 = O$ and the initial position T_0 of the tamer is at distance $|OT_0| = 1/2$ from the centre. We now specify the tamer's locations T_k inductively for all $k \geq 1$. Given T_{k-1} for some $k \geq 1$ we require T_k to be such that $T_{k-1}T_k$ is perpendicular to OT_{k-1}, with L_{k-1} and T_k lying on opposite sides of the straight line through O and T_{k-1}. It may help to draw a picture! If it happens that L_{k-1} lies exactly *on* the line through O and T_{k-1} then we may chose either of the two possible directions without affecting the arguments that follow. To complete our description of the tamer's strategy we only need to specify the distances $|T_{k-1}T_k|$, $k \geq 1$, travelled at each stage. Before we do this, let us observe that no matter what distances we choose the tamer cannot be caught at any stage while on the path $T_0T_1T_2\dots$. This follows easily from the fact that the hypotenuse is always the longest side in a right-angled triangle. Now what conditions should the distances $|T_{k-1}T_k|$, $k \geq 1$, satisfy? Well, for one thing we need to ensure that the tamer remains within the arena at all times, which amounts to requiring $|OT_k| \leq 1$ for all $k \geq 1$. Using Pythagoras' theorem it is not difficult to see that $|OT_k|^2 = \frac{1}{4} + |T_0T_1|^2 + \cdots + |T_{k-1}T_k|^2$ for $k \geq 1$. So what we need is that

$$\sum_{k=1}^{\infty} |T_{k-1}T_k|^2 \leq \frac{3}{4}. \tag{1}$$

In order for the tamer to escape capture indefinitely, we also need the total length of the path $T_0T_1T_2T_3\ldots$ (and hence the duration of the chase) to be infinite, that is

$$\sum_{k=1}^{\infty} |T_{k-1}T_k| = \infty. \tag{2}$$

The question now is: can we choose the distances $|T_{k-1}T_k|$, $k \geq 1$, in such a way that (1) and (2) are satisfied simultaneously? We can, provided we know just a little about convergent and divergent series. It can be shown, for instance, that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Knowing this we can check that both (1) and (2) hold for

$$|T_{k-1}T_k| = \frac{3}{\sqrt{2}\pi k}, \quad k \geq 1.$$

There are many other choices of the distances $|T_{k-1}T_k|$, $k \geq 1$, which also work. For example, we could choose $|T_{k-1}T_k| = ck^{-1}$ for any constant c such that $0 < c \leq 3/\sqrt{2}\pi$, and knowing just a little bit more about the convergence and divergence of series we see that $|T_{k-1}T_k| = ck^{-\alpha}$ works, too, provided $1/2 < \alpha \leq 1$ and c is a sufficiently small positive constant. And of course some of you may have come up with a different strategy altogether! If you liked this problem you might also enjoy having a look at the book *Chases and Escapes: The Mathematics of Pursuit and Evasion* by P.J. Nahin, which includes more background information as well as a large number of broadly similar puzzles.